On the number of SDRs of a valued (t, n)-family*

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Abstract

A system of distinct representatives (SDR) of a family $F = (A_1, \cdots, A_n)$ is a sequence $(x_1, \cdots, x_n)$ of n distinct elements with $x_i \in A_i$ for $1 \le i \le n$. Let $N(F)$ denote the number of SDRs of a family F; two SDRs are considered distinct if they are different in at least one component. For a nonnegative integer t, a family $F = (A_1, \cdots, A_n)$ is called a (t, n)-family if the union of any $k \ge 1$ sets in the family contains at least $k + t$ elements. The famous Hall's Theorem says that $N(F) \ge 1$ if and only if F is a $(0, n)$-family. Denote by $M(t, n)$ the minimum number of SDRs in a (t, n)-family. The problem of determining $M(t, n)$ and those families containing exactly $M(t, n)$ SDRs was first raised by Chang [European J. Combin.**10**(1989), 231-234]. He solved the cases when $0 \le t \le 2$ and gave a conjecture for $t \ge 3$. In this paper, we solve the conjecture. In fact, we get a more general result for so-called valued (t, n)-family.

Keywords. A system of distinct representatives, Hall's Theorem, (t, n)-family.

1 Introduction

A system of distinct representatives (SDR) of a family $F = (A_1, \cdots, A_n)$ is a sequence $(x_1, \cdots, x_n)$ of n distinct elements with $x_i \in A_i$ for $1 \le i \le n$. The famous Hall's theorem [4] tell us that a family has a SDR if and only if the union of any $k \ge 1$ sets of this family contains at least k elements. Several quantative refinements of the Hall's theorem were given in [3, 6, 7]. Their results are all under the assumption of Hall's condition plus some extra conditions on the cardinalities of A_i's.

Chang [1] extends Hall's theorem as follows: let t be a nonnegative integer. A family $F = (A_1, \cdots, A_n)$ is called a (t, n)-*family* if $|\bigcup_{i \in I} A_i| \ge |I| + t$ holds for any non-empty subset $I \subseteq \{1, \cdots, n\}$. Denote by $N(F)$ the number of SDRs of a family F. Let $M(t, n) = \min\{N(F) \mid F \text{ is a } (t, n)\text{-family}\}$. Hall's theorem says that $M(0, n) \ge 1$. In fact, it is easy to know that $M(0, n) = 1$. Chang [1] proved that $M(1, n) = n+1$ and $M(2, n) = n^2+n+1$. He also determined all (t, n)-families F with $N(F) = M(t, n)$ for $t = 0, 1, 2$. Consider the (t, n)-family $F^* = (A_1^*, \cdots, A_n^*)$, where $A_i^* = \{i, n+1, \cdots, n+t\}$ for $1 \le i \le n$. Then,

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$$N(F^*) = U(t,n) = \sum_{j=0}^{t} \binom{t}{j}\binom{n}{j} j!.$$

Chang[1] has shown that F^* as above is the only $(2,n)$-family F with $N(F) = M(t,n)$, and he conjectured that $M(t,n) = U(t,n)$ and F^* is the only (t,n)-family F with $N(F) = M(t,n)$ for all $t \geq 3$. In 1992, Leung and Wei [5] claimed that they proved the above conjecture by means of a comparison theorem for permanents. But Leung and Wei's proof has a fatal mistake (see [2]). Hence, the conjecture is still open. In this paper, we solve the conjecture. In fact, we get a more general result for so-called valued (t,n)-family. In what follow, we assume that $t \geq 2$.

For a sequence of positive integers $(a_1, \cdots, a_n)$, a family $F = (A_1, \cdots, A_n)$ is called a *valued (t,n)-family with valuation* $(a_1, \cdots, a_n)$ if $|A_i| = a_i + t$ and $|\bigcup_{i \in I} A_i| \geq \sum_{i \in I} a_i + t$ for any $|I| \geq 2$. Note that a (t,n)-family $F = (A_1, \cdots, A_n)$ with $N(F) = M(t,n)$ must have $|A_i| = t+1$ for $1 \leq i \leq n$ (see Lemmas 1 and 2 in [1]). Hence, a (t,n)-family F with $N(F) = M(t,n)$ is a valued (t,n)-family with valuation $(1, \cdots, 1)$. Let $\bar{F}$ be a valued (t,n)-family with valuation $(a_1, \cdots, a_n)$ satisfying $|\bigcap_{i \in I} A_i| = t$ for any $|I| \geq 2$. Hence, F^* is $\bar{F}$ with valuation $(1, \cdots, 1)$. Define $M'(t,n,a_1,\cdots,a_n) = \min\{N(F) \mid F$ is a valued (t,n)-family with valuation $(a_1, \cdots, a_n)\}$, and let

$$U'(t,n,a_1,\cdots,a_n) = N(\bar{F}) = \sum_{j=0}^{t}\left[\binom{t}{j} j! \sum_{1 \leq i_1 < \cdots < i_{n-j} \leq n} a_{i_1} \cdots a_{i_{n-j}}\right].$$

In this paper, we will prove that $M'(t,n,a_1,\cdots,a_n) = U'(t,n,a_1,\cdots,a_n)$ and $\bar{F}$ is the only valued (t,n)-family F with valuation $(a_1,\cdots,a_n)$ satisfying $N(F) = M'(t,n,a_1,\cdots,a_n)$ for $t \geq 2$. The conjecture of Chang [1] is a direct corollary of the conclusion.

Some notations are needed. Suppose F is a valued (t,n)-family with valuation $(a_1,\cdots,a_n)$. Let $N = \{1,2,\cdots,n\}$ and $\mathcal{B} = \bigcup_{i \in N} A_i$, and let $I_x = \{i \in N \mid x \in A_i\}$ and $I_x^c = N - I_x$ for $x \in \mathcal{B}$. The *degree of* x, denoted by $\deg x$, is $|I_x|$. A pair of elements $\{x,y\} \subseteq \mathcal{B}$ is *exclusive* if $I_x \cap I_y^c \neq \emptyset$ and $I_y \cap I_x^c \neq \emptyset$. An exclusive pair $\{x,y\}$ is *saturated* if there exists a subset $I \subseteq N$ satisfying $I \cap I_x \cap I_y = \emptyset$, $I \cap I_x \cap I_y^c \neq \emptyset$, $I \cap I_x^c \cap I_y \neq \emptyset$ and $|\bigcup_{i \in I} A_i| = \sum_{i \in I} a_i + t$; otherwise, we say an exclusive pair $\{x,y\}$ is *unsaturated.*

2 An exclusive pair $\{x,y\}$ for a valued (t,n)-family

Assume that $F = (A_1,\cdots,A_n)$ is a valued (t,n)-family with valuation $(a_1,\cdots,a_n)$ and a pair of elements $\{x,y\}$ is exclusive for F. Let

$$A_i(x,y) = \begin{cases} A_i - \{x\} \cup \{y\}, & i \in I_x \cap I_y^c; \\ A_i, & \text{otherwise.} \end{cases}$$

Then we get a new family $F_y^x = (A_1(x,y),\cdots,A_n(x,y))$, but it is possible that F_y^x is not a valued (t,n)-family with valuation $(a_1,\cdots,a_n)$. For any $I \subseteq N$, by calculating $|\bigcup_{i \in I} A_i|$ and $|\bigcup_{i \in I} A_i(x,y)|$, we can get the relationship between the two values as follows:

$$|\bigcup_{i\in I} A_i(x,y)| = \begin{cases} |\bigcup_{i\in I} A_i| - 1, & I\cap I_x\cap I_y = \emptyset, I\cap I_x\cap I_y^c \neq \emptyset, I\cap I_x^c\cap I_y \neq \emptyset; \\ |\bigcup_{i\in I} A_i|, & \text{otherwise.} \end{cases}$$

Hence, F_y^x is also a valued (t,n)-family with valuation $(a_1,\cdots,a_n)$ if and only if $\{x,y\}$ is unsaturated for F. Furthermore, we have

Theorem 1 *A valued (t,n)-family with valuation $(a_1,\cdots,a_n)$ satisfying $N(F) = M'(t,n,a_1,\cdots,a_n)$ does not contain any unsaturated pair $\{x,y\}$.*

Proof. Suppose to the contrary that $\{x,y\}$ is unsaturated for F. Then, F_y^x is also a valued (t,n)-family with valuation $(a_1,\cdots,a_n)$. We will prove that $N(F_y^x) < N(F)$ and hence leads to a contradiction.

Without lose of generality, we can assume that $I_x\cap I_y^c = \{1,\cdots,k_1\} \neq \emptyset$, $I_y\cap I_x^c = \{k_1+1,\cdots,k_2\} \neq \emptyset$, $I_x\cap I_y = \{k_2+1,\cdots,k_3\}$ and $I_x^c\cap I_y^c = \{k_3+1,\cdots,n\}$. So $F_y^x = (A_1(x,y),\cdots,A_n(x,y)) = (A_1-\{x\}\cup\{y\},\cdots,A_{k_1}-\{x\}\cup\{y\},A_{k_1+1},\cdots,A_n)$. Let $(x_1,\cdots,x_n)$ be an SDR of F_y^x. Define a function f from the set of all SDRs of F_y^x to the set of all SDRs of F as follows:

(a) if $x_i = y$ for some $i\in\{1,\cdots,k_1\}$ and $x_j = x$ for some $j\in\{k_2+1,\cdots,k_3\}$, then

$$(x_1,\cdots,y,\cdots,x,\cdots,x_n) \to (x_1,\cdots,x,\cdots,y,\cdots,x_n).$$

(b) if $x_i = y$ for some $i\in\{1,\cdots,k_1\}$ and $x_j \neq x$ for all x_j, then

$$(x_1,\cdots,y,\cdots,x_n) \to (x_1,\cdots,x,\cdots,x_n).$$

(c) otherwise,

$$(x_1,\cdots,x_n) \to (x_1,\cdots,x_n).$$

f is clearly one to one. Define

$$F' = (A_2-\{x,y\},\cdots,A_{k_1}-\{x,y\},A_{k_1+2}-\{x,y\},\cdots,A_n-\{x,y\}).$$

When $t\geq 2$, F' satisfies the Hall's condition and has an SDR $(x_2,\cdots,x_{k_1},x_{k_1+2},\cdots,x_n)$. Hence, F has an SDR such as

$$(x,x_2,\cdots,x_{k_1},y,x_{k_1+2},\cdots,x_n),$$

which is not an f-image of an SDR of F_y^x, so f is not subjective. Hence, $N(F_y^x) < N(F)$.

3 Saturated pairs of a valued $(t,n)-$family

For the set $N = \{1,\cdots,n\}$, we define a relation "$\sim$" on N as follows: $i\sim j$ if and only if there exists a subset I satisfying $\{i,j\}\subseteq I\subseteq N$ and $|\bigcup_{s\in I} A_s| = \sum_{s\in I} a_s + t$. We claim that "$\sim$" is an equivalent relation on N. It is obvious that "$\sim$" is reflexive and symmetric. If $i\sim j$ and $j\sim k$, then there exist I and J satisfying $\{i,j\}\subseteq I$, $|\bigcup_{s\in I} A_s| = \sum_{s\in I} a_s + t$ and $\{j,k\}\subseteq J$, $|\bigcup_{s\in J} A_s| = \sum_{s\in J} a_s + t$, respectively. Note that $I\cap J\neq\emptyset$ as $j\in I\cap J$. Hence, we have

$$\begin{aligned}\sum_{s\in I\cup J} a_s + t &\leq |\bigcup_{s\in I\cup J} A_s| = |(\bigcup_{s\in I} A_s)\cup(\bigcup_{s\in J} A_s)| \\ &\leq |\bigcup_{s\in I} A_s| + |\bigcup_{s\in J} A_s| - |\bigcup_{s\in I\cap J} A_s| \\ &\leq \sum_{s\in I} a_s + t + \sum_{s\in J} a_s + t - (\sum_{s\in I\cap J} a_s + t) \\ &= \sum_{s\in I\cup J} a_s + t.\end{aligned}$$

So we know that $|\bigcup_{s\in I\cup J} A_s| = \sum_{s\in I\cup J} a_s + t$ and $\{i,k\} \subseteq I\cup J$. It implies that $i \sim k$ and "$\sim$" is transitive. Hence, "$\sim$" is an equivalent relation. So we can classify N into different classes: $C_1, \cdots, C_m$. If an index set $I \subseteq N$ satisfies $|\bigcup_{i\in I} A_i| = \sum_{i\in I} a_i + t$, by the definition of "$\sim$", we know that $I \subseteq C_i$ for some $i \in \{1, \cdots, m\}$.

Theorem 2 *For a valued (t,n)-family F with valuation $(a_1, \cdots, a_n)$, denote by $NSP(F)$ the number of saturated pairs of F, then $NSP(F) \leq \sum_{1\leq i<j\leq n} a_i a_j$.*

Proof. We use induction on n. When $n = 2$, the conclusion is obvious.

If $|\mathcal{B}| > \sum_{i=1}^{n} a_i + t$, then by the classification of N under the equivalent relation "$\sim$", we get several classes $C_1, \cdots, C_m$ and $m \geq 2$. Without lose of generality, we can assume that $C_1 = \{1, \cdots, k_1\}, \cdots, C_m = \{k_{m-1}+1, \ldots, n\}$. We get m subfamilies $F_1, \cdots, F_m$ with index sets $C_1, \cdots, C_m$, respectively. According to the preparation before Theorem 2, we know that each saturated pair of F must be saturated for some subfamily F_i. Hence, $NSP(F) \leq NSP(F_1) + \cdots + NSP(F_m)$. By induction,

$$NSP(F) \leq \sum_{1\leq i<j\leq k_1} a_i a_j + \cdots + \sum_{k_{m-1}+1\leq i<j\leq n} a_i a_j < \sum_{1\leq i<j\leq n} a_i a_j.$$

Now we assume that $|\mathcal{B}| = \sum_{i=1}^{n} a_i + t$. Let I be an index set satisfying the following conditions: (1) $|I| \geq 2$; (2) $|\bigcup_{i\in I} A_i| = \sum_{i\in I} a_i + t$; (3) For $J \subset I$, if $|J| \geq 2$, then $|\bigcup_{i\in J} A_i| > \sum_{i\in J} a_i + t$. Since $|\mathcal{B}| = \sum_{i=1}^{n} a_i + t$, the existence of such I holds. Now we use different methods to discuss two cases $I \subset N$ and $I = N$.

For $I \subset N$, without lose of generality, we can assume that $I = \{k+1, \cdots, n\}, k \geq 1$. Let $B_1 = A_1, \ldots, B_k = A_k, B_{k+1} = \bigcup_{i=k+1}^{n} A_i$, then $G = (B_1, \cdots, B_{k+1})$ is a valued $(t, k+1)$-family with valuation $(a_1, \cdots, a_k, \sum_{i=k+1}^{n} a_i)$. Let $\{x,y\}$ be an arbitrary saturated pair for F. There are three subcases: (1) $\{x,y\}$ is saturated for the subfamily $(A_1, \cdots, A_k)$; (2) $\{x,y\}$ is saturated for the subfamily $(A_{k+1}, \cdots, A_n)$; (3) $\{x,y\}$ is unsaturated for both $(A_1, \cdots, A_k)$ and $(A_{k+1}, \cdots, A_n)$. It is easy to see that $\{x,y\}$ in the subcase (1) is also saturated for the family G.

We claim that $\{x,y\}$ in the subcase (3) is also saturated for G. Since $\{x,y\}$ is saturated for F and unsaturated for both $(A_1,\cdots,A_k)$ and $(A_{k+1},\cdots,A_n)$, there exist $\emptyset\neq I_1\subseteq\{1,\cdots,k\}$ and $\emptyset\neq I_2\subseteq I=\{k+1,\cdots,n\}$ such that $|\bigcup_{i\in I_1\cup I_2}A_i|=\sum_{i\in I_1\cup I_2}a_i+t$ and $(I_1\cup I_2)\cap I_x\cap I_y=\emptyset$, $(I_1\cup I_2)\cap I_x\cap I_y^c\neq\emptyset$, $(I_1\cup I_2)\cap I_y\cap I_x^c\neq\emptyset$. Since $|\bigcup_{i\in I_1\cup I_2}A_i|=\sum_{i\in I_1\cup I_2}a_i+t$ and $|\bigcup_{i\in I}A_i|=\sum_{i=k+1}^{n}a_i+t$, using the same discussion in the proof of transitivity of " $\sim''$, we can show that $|(\bigcup_{i\in I_1}B_i)\cup B_{k+1}|=|(\bigcup_{i\in I_1}A_i)\cup(\bigcup_{i\in I}A_i)|=|(\bigcup_{i\in I_1\cup I_2}A_i)\cup(\bigcup_{i\in I}A_i)|=\sum_{i\in I_1}a_i+\sum_{i=k+1}^{n}a_i+t$. Under these circumstances, if $\{x,y\}$ is not a subset of B_{k+1}, then $\{x,y\}$ is saturated for G.

Now we will prove that $\{x,y\}$ is not a subset of B_{k+1} in two cases: $|I_2|\geq 2$ and $|I_2|=1$.

If $|I_2|\geq 2$, we claim that $I_2=I$. Suppose to the contrary that $I_2\subset I$. According to $I=\{k+1,\cdots,n\}$, we know that $|\bigcup_{i\in I}A_i|=\sum_{i=k+1}^{n}a_i+t$ and $|\bigcup_{i\in I_2}A_i|>\sum_{i\in I_2}a_i+t$. So $|(\bigcup_{i\in I}A_i)-(\bigcup_{i\in I_2}A_i)|<\sum_{i=k+1}^{n}a_i-\sum_{i\in I_2}a_i$. Hence,

$$\begin{aligned}
|\bigcup_{i\in I_1\cup I}A_i| &= |\bigcup_{i\in I_1\cup I_2}A_i|+|(\bigcup_{i\in I-I_2}A_i)-(\bigcup_{i\in I_1\cup I_2}A_i)|\\
&\leq |\bigcup_{i\in I_1\cup I_2}A_i|+|(\bigcup_{i\in I}A_i)-(\bigcup_{i\in I_2}A_i)|\\
&< \sum_{i\in I_1\cup I_2}a_i+t+\sum_{i=k+1}^{n}a_i-\sum_{i\in I_2}a_i\\
&= \sum_{i\in I_1\cup I}a_i+t.
\end{aligned}$$

It contradicts with the fact that F is a valued (t,n)-family with valuation $(a_1,\cdots,a_n)$. Hence, $I_2=I$.

Now we know that $(I_1\cup I)\cap I_x\cap I_y=\emptyset$, and hence $I\cap I_x\cap I_y=\emptyset$. Since $|\bigcup_{i\in I}A_i|=\sum_{i\in I}a_i+t$ and $\{x,y\}$ is unsaturated for the subfamily $(A_{k+1},\cdots,A_n)$, we have either $I\cap I_x\cap I_y^c=\emptyset$ or $I\cap I_x^c\cap I_y=\emptyset$. Furthermore, we have either $I\cap I_x=\emptyset$ or $I\cap I_y=\emptyset$. Therefore, $B_{k+1}=\bigcup_{i\in I}A_i$ contains at most one of x,y, so $\{x,y\}$ is not a subset of B_{k+1}.

If $|I_2|=1$, without lose of generality, we can assume that $I_2=\{k+1\}$. Since $(I_1\cup I_2)\cap I_x\cap I_y=\emptyset$, we know that $k+1\notin I_x\cap I_y$, which implies that A_{k+1} contains at most one of x,y. Assume that $y\notin A_{k+1}$. Suppose to the contrary that $\{x,y\}$ is a subset of B_{k+1}, then $y\in\bigcup_{i\in I-I_2}A_i$. By the selection of I_1 and I_2, we know that $y\in\bigcup_{i\in I_1\cup I_2}A_i$, and hence $y\notin(\bigcup_{i\in I-I_2}A_i)-(\bigcup_{i\in I_1\cup I_2}A_i)$. Then,

$$|(\bigcup_{i\in I-I_2}A_i)-(\bigcup_{i\in I_1\cup I_2}A_i)|<|(\bigcup_{i\in I}A_i)-A_{k+1}|.$$

Since $|A_{k+1}|=a_{k+1}+t$ and $|\bigcup_{i\in I}A_i|=\sum_{i=k+1}^{n}a_i+t$, we know that

$$|(\bigcup_{i\in I} A_i) - A_{k+1}| = |\bigcup_{i\in I} A_i| - |A_{k+1}| = \sum_{i=k+2}^{n} a_i.$$

Therefore,

$$\begin{aligned}
|(\bigcup_{i\in I_1} A_i) \cup (\bigcup_{i\in I} A_i)| &= |(\bigcup_{i\in I_1} A_i) \cup A_{k+1} \cup (\bigcup_{i\in I-I_2} A_i)| \\
&= |(\bigcup_{i\in I_1} A_i) \cup A_{k+1}| + |(\bigcup_{i\in I-I_2} A_i) - (\bigcup_{i\in I_1\cup I_2} A_i)| \\
&= \sum_{i\in I_1} a_i + a_{k+1} + t + |(\bigcup_{i\in I-I_2} A_i) - (\bigcup_{i\in I_1\cup I_2} A_i)| \\
&< \sum_{i\in I_1} a_i + \sum_{i=k+1}^{n} a_i + t
\end{aligned}$$

This contradicts with the fact that F is a valued (t,n)-family with valuation $(a_1,\cdots,a_n)$. Hence, $\{x,y\}$ is not a subset of B_{k+1}.

Now we have shown that when $I \subset N$, any saturated pair $\{x,y\}$ for F is saturated for either G or the subfamily $(A_{k+1},\cdots,A_n)$. Therefore,

$$NSP(F) \leq NSP(G) + NSP((A_{k+1},\cdots,A_n))$$

by induction, we have

$$NSP(G) \leq \sum_{1\leq i<j\leq k} a_i a_j + (\sum_{l=1}^{k} a_l)(\sum_{m=k+1}^{n} a_m)$$

and

$$NSP((A_{k+1},\cdots,A_n)) \leq \sum_{k+1\leq i<j\leq n} a_i a_j$$

Hence, $NSP(F) \leq \sum\limits_{1\leq i<j\leq n} a_i a_j$.

When $I = N$, an exclusive pair $\{x,y\}$ is saturated for F if and only if $I_x \cap I_y = \emptyset$. Let $C = \{\{x,y\} \mid I_x \cap I_y = \emptyset\}$. Then $NSP(F) = |C|$. Now we calculate $|C|$.

For an arbitrary element $z \in \mathcal{B}$, define $C(z) = \{\{x,z\} \mid I_x \cap I_z = \emptyset\}$. It is not difficult to see that $|C| = \frac{1}{2}\sum\limits_{z\in\mathcal{B}} |C(z)|$ and $C(z) = \{\{x,z\} \mid I_x \cap I_z = \emptyset\} = \{\{x,z\} \mid x \notin \bigcup\limits_{i\in I_z} A_i\}$. So,

$$|C(z)| = |\mathcal{B}| - |\bigcup_{i\in I_z} A_i| \leq \sum_{i\in I_z^c} a_i.$$

Therefore,

$$\begin{aligned}
|C| &\leq \frac{\sum\limits_{z\in\mathcal{B}}\sum\limits_{i\in I_z^c} a_i}{2} = \frac{\sum\limits_{z\in\mathcal{B}}(\sum\limits_{i=1}^{n} a_i - \sum\limits_{i\in I_z} a_i)}{2} \\
&= \frac{(\sum\limits_{i=1}^{n} a_i + t)(\sum\limits_{i=1}^{n} a_i) - \sum\limits_{z\in\mathcal{B}}\sum\limits_{i\in I_z} a_i}{2}
\end{aligned}$$

$$= \frac{(\sum_{i=1}^{n} a_i + t)(\sum_{i=1}^{n} a_i) - \sum_{i=1}^{n}(a_i + t)a_i}{2}$$
$$= \sum_{1 \le i < j \le n} a_i a_j.$$

4 Exclusive pairs of a valued (t, n)-family

Theorem 3 *For a valued (t, n)-family F with valuation $(a_1, \cdots, a_n)$, denote by $NEP(F)$ the number of exclusive pairs of F, then $NEP(F) \ge \sum_{1 \le i<j \le n} a_i a_j$. $\bar{F}$ is the only valued (t, n)-family F with valuation $(a_1, \cdots, a_n)$ satisfying $NEP(F) = \sum_{1 \le i<j \le n} a_i a_j$.*

Proof. We can assume that $n \ge 2$. For an arbitrary element $z \in \mathcal{B}$, $\{x, z\}$ is exclusive for F if and only if $x \in \bigcup_{i \in I_z^c} A_i$ and $x \notin \bigcap_{i \in I_z} A_i$. Define $D(z) = \{\{x, z\} \mid \{x, z\}$ is exclusive for $F\}$. Therefore,

$$D(z) = \{\{x, z\} \mid x \in \bigcup_{i \in I_z^c} A_i - \bigcap_{i \in I_z} A_i\}.$$

Let $\mathcal{A} = \{z | \deg z = n\}$ and $D = \{\{x, y\} \mid \{x, y\}$ is exclusive for $F\}$. Note that $D(z) = \emptyset$ if $z \in \mathcal{A}$. Then,

$$\begin{aligned} |D| &= \frac{1}{2} \sum_{z \in \mathcal{B}} |D(z)| = \frac{1}{2} \sum_{z \in \mathcal{B} - \mathcal{A}} |D(z)| \\ &= \frac{1}{2} \sum_{z \in \mathcal{B}-\mathcal{A}} (|\bigcup_{i \in I_z^c} A_i - \bigcap_{i \in I_z} A_i|). \end{aligned}$$

We first assume that $\deg z \ge 2$ for all $z \in \mathcal{B} - \mathcal{A}$. Then $|I_z| \ge 2$ and hence $|\bigcap_{i \in I_z} A_i| \le t$ for all $z \in \mathcal{B} - \mathcal{A}$. Hence,

$$|D| > \frac{1}{2} \sum_{z \in \mathcal{B}-\mathcal{A}} (|\bigcup_{i \in I_z^c} A_i| - |\bigcap_{i \in I_z} A_i|) \ge \frac{1}{2} \sum_{z \in \mathcal{B}-\mathcal{A}} \sum_{i \in I_z^c} a_i. \qquad (*)$$

We point out that the inequality strictly holds as $z \in \bigcap_{i \in I_z} A_i$ and $z \notin \bigcup_{i \in I_z^c} A_i$. To calculate $\sum_{z \in \mathcal{B}-\mathcal{A}} \sum_{i \in I_z^c} a_i$, we construct a weighted bipartite graph G as follows: $V(G) = V_1 \cup V_2$, where $V_1 = \mathcal{B} - \mathcal{A}$ and $V_2 = \{A_1, \cdots, A_n\}$; For $z \in V_1$, if $z \notin A_i$, then $zA_i \in E(G)$ and the weight of zA_i, denoted by $w(zA_i)$, is a_i. So,

$$\sum_{z \in \mathcal{B}-\mathcal{A}} \sum_{i \in I_z^c} a_i = \sum_{z \in V_1} \sum_{zA_i \in E(G)} w(zA_i) = \sum_{A_i \in V_2} \sum_{zA_i \in E(G)} w(zA_i). \qquad (**)$$

Let $|\mathcal{A}| = a$. Obviously, $a \le t$. Each set A_i contains $a_i + t - a$ elements in $\mathcal{B} - \mathcal{A}$ and there are at least $\sum_{j=1}^{n} a_j + t - a$ elements in $\mathcal{B} - \mathcal{A}$. By the construction of G, we know

that the vertex A_i is incident to at least $\sum\limits_{j=1}^{n} a_j - a_i$ edges in G and the weight of each edge incident to A_i is a_i. Therefore,

$$\sum_{A_i\in V_2}\sum_{zA_i\in E(G)} w(zA_i) \geq \sum_{i=1}^{n} a_i(\sum_{j=1}^{n} a_j - a_i) = (\sum_{i=1}^{n} a_i)^2 - \sum_{i=1}^{n} a_i^2. \qquad (***)$$

By above inequalities $(*)$, $(**)$ and $(***)$, we know that $|D| > \sum\limits_{1\leq i<j\leq n} a_i a_j$ if $\deg z \geq 2$ for all $z \in \mathcal{B}$.

Now we assume that there exists an element x such that $\deg x = 1$, without lose of generality, we assume that $I_x = \{n\}$. Let $k = \sum\limits_{i=1}^{n} a_i$. We use induction on k.

When $k = 2$, then $n = 2$ and $a_1 = a_2 = 1$, the conclusion is obvious. Assume that $k \geq 3$. As the conclusion is obvious when $n = 2$, we may assume that $n \geq 3$.

If $a_n = 1$, let $F_1 = (A_1, \cdots, A_{n-1})$, by induction hypothesis, $NEP(F_1) \geq \sum\limits_{1\leq i<j\leq n-1} a_i a_j$ and $NEP(F_1) = \sum\limits_{1\leq i<j\leq n-1} a_i a_j$ implies that F_1 is $\bar{F}$ with valuation $(a_1, \cdots, a_{n-1})$. It is obvious that the exclusive pairs of F_1 are also exclusive for F. Since $(\bigcup\limits_{i=1}^{n-1} A_i) - A_n = (\bigcup\limits_{i=1}^{n} A_i) - A_n$, we know that $|\bigcup\limits_{i=1}^{n-1} A_i - A_n| \geq \sum\limits_{i=1}^{n-1} a_i$. Obviously, each element y in $(\bigcup\limits_{i=1}^{n-1} A_i) - A_n$ is exclusive with x for F and $\{x, y\}$ is different from any exclusive pair of $(A_1, \cdots, A_{n-1})$. Therefore,

$$NEP(F) \geq \sum_{1\leq i<j\leq n-1} a_i a_j + \sum_{k=1}^{n-1} a_k = \sum_{1\leq i<j\leq n} a_i a_j.$$

When $NEP(F) = \sum\limits_{1\leq i<j\leq n} a_i a_j$, it implies that $A_n \cap (\bigcup\limits_{i=1}^{n-1} A_i) = t$ and $NEP(F) - NEP(F_1) = \sum\limits_{k=1}^{n-1} a_k$. This requires that F is $\bar{F}$ with valuation $(a_1, \cdots, a_n)$.

If $a_n \geq 2$, let $F_2 = (A_1, \cdots, A_{n-1}, A_n - \{x\})$, which is a (t, n)-family with valuation $(a_1, \cdots, a_{n-1}, a_n - 1)$, by induction hypothesis, $NEP(F_2) \geq \sum\limits_{1\leq i<j\leq n-1} a_i a_j + \sum\limits_{k=1}^{n-1} a_k(a_n - 1)$ and $NEP(F_2) = \sum\limits_{1\leq i<j\leq n-1} a_i a_j + \sum\limits_{k=1}^{n-1} a_k(a_n - 1)$ implies that F_2 is $\bar{F}$ with valuation $(a_1, \cdots, a_{n-1}, a_n - 1)$. Similarly, the exclusive pairs of F_2 are also exclusive for F, $|\bigcup\limits_{i=1}^{n-1} A_i - A_n| \geq \sum\limits_{i=1}^{n-1} a_i$, and each element y in $\bigcup\limits_{i=1}^{n-1} A_i - A_n$ is exclusive with x for F and $\{x, y\}$ is different from any exclusive pair of F_2. Therefore,

$$NEP(F) \geq \sum_{1\leq i<j\leq n-1} a_i a_j + \sum_{k=1}^{n-1} a_k(a_n - 1) + \sum_{k=1}^{n-1} a_k = \sum_{1\leq i<j\leq n} a_i a_j.$$

Similarly, when $NEP(F) = \sum\limits_{1\leq i<j\leq n} a_i a_j$, it implies that F_2 must be $\bar{F}$ with valuation $(a_1, \cdots, a_{n-1}, a_n - 1)$, and since $I_x = \{n\}$, it is obvious that F is $\bar{F}$ with valuation $(a_1, \cdots, a_n)$.

5 The conclusion about $N(F)$

By Theorem 1, 2 and 3, we can easily arrive at the following conclusion:

Theorem 4 *$M^{'}(t, n, a_1, \cdots, a_n) = U^{'}(t, n, a_1, \cdots, a_n)$ and $\bar{F}$ is the only valued (t, n)-family F with valuation $(a_1, \cdots, a_n)$ satisfying $N(F) = M^{'}(t, n, a_1, \cdots, a_n)$ for $t \geq 2$.*

Applying Theorem 4 to (t, n)-family, we immediately prove the conjecture of Chang in [1].

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